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On A Sturm - Liouville Like Four Point Boundary Value Problem

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I. Introduction

In this article we consider the problem

$$
\begin{aligned}
&x''(t) + h(t)f(t, x(t), x'(t)) = 0, \quad t \in [0, 1], \\
&x'(0) - \alpha_1 x(\xi) = 0, \quad x'(1) + \alpha_2 x(\eta) = 0,
\end{aligned}
\tag{1.1}
$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\xi \in (0, 1)$, $\eta \in (0, 1)$, $\xi \neq \eta$, $h(t) \in \mathcal{C}(\mathbb{R})$, $f(\cdot, \cdot, \cdot) \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ are fixed, $x(t)$ is unknown..

Our aim is to investigate the problem (1.1) for existence of solutions. For this purpose we propose new approach for investigation. This approach gives new results.

Our main result is as follows.

Theorem 1.1. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\xi \in (0, 1)$, $\eta \in (0, 1)$, $\xi \neq \eta$, $h(t) \in \mathcal{C}(\mathbb{R})$, $f(\cdot, \cdot, \cdot) \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ be fixed. Then*

1) *the problem (1.1) has a bounded solution $x(t) \in \mathcal{C}^2([0, 1])$;*

2) *if for the bounded solution $x(t)$ of 1) we have*

$$
\int_\eta^t \int_0^s h(\tau) f(\tau, x(\tau), x'(\tau)) d\tau ds \neq 0 \quad \text{some} \quad t \in [0, 1],
$$

then it doesn't coincide with zero on $[0, 1]$,

3) *if for the bounded solution $x(t)$ of 1) we have*

$$
h(t) f(t, x(t), x'(t)) \neq 0 \quad \text{for} \quad \text{some} \quad t \in [0, 1],
$$

then it doesn't coincide with a constant.

We will compare our result with well known result.

In [1] the problem (1.1) is considered under conditions $0 < \alpha_1 < \frac{1}{\xi}$, $0 < \alpha_2 < \frac{1}{1-\eta}$, $0 < \xi < \eta < 1$, $\alpha_1 \alpha_2 \eta - \alpha_1 \alpha_2 \xi + \alpha_1 + \alpha_2 > 0$, $h(t) : [0, 1] \longrightarrow [0, \infty)$ is a continuous function,

Ref.
[1] Zhao, J., F. Geng, J. Zhao, W. Ge. Positive solutions for a new kind Sturm - Liouville like foour point boundary value problem, Applied Mathematics and Computations, 2010, pp.811-819.

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a.e. $t \in [0,1]$, $f(t,x,y) \leq a(t) + b(t)x + c(t)y$ for suitable functions $a, b, c \in L^1([0,1])$ and it is proved that (1.1) has a nontrivial solution. Evidently our result is better than the result in [1].

II. Proof of Main Result

1) Let D_1 be fixed positive constant and let also

$$M_1 = \max\Big\{\max_{t\in[0,1]} |h(t)|, \max_{[0,1]\times[-D_1,D_1]\times[-D_1,D_1]} |f(\cdot,\cdot,\cdot)|\Big\}.$$

Let $a_1 \in (0,1)$ is enough closed to 1 and $\epsilon \in (0,1)$ are chosen so that $a_1 + \epsilon_1 > 1$ and

$$\begin{aligned} &\epsilon_1 D_1 + (1-a_1)\tfrac{D_1}{|\alpha_2|} + (1-a_1)|\alpha_1| D_1 + (1-a_1)M_1 \leq D_1, \\ &\epsilon_1 D_1 + (1-a_1)|\alpha_1| D_1 + (1-a_1)M_1 \leq D_1. \end{aligned} \tag{2.1}$$

We define the sets

$$N_1 = \Big\{x(t) \in \mathcal{C}^1([0,1]) : |x(t)| \leq D_1, |x'(t)| \leq D_1 \quad \forall t \in [0,1]\Big\},$$

$$N_1^* = \Big\{x(t) \in \mathcal{C}^1([0,1]) : |x(t)| \leq (a_1+\epsilon_1)D_1, |x'(t)| \leq (a_1+\epsilon_1)D_1 \quad \forall t \in [0,1]\Big\}.$$

In these sets we define a norm as follows $||x|| = \max\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |x'(t)|\}$. With this norm the sets N_1 and N_1^* are completely normed spaces. Also since for $x \in N_1$ we have $|x(t)| \leq D_1$, $|x'(t)| \leq D_1$ for every $t \in [0,1]$ we have that N_1 is a compact subset and closed convex subset of N_1^*.

Under these sets we define the operators

$P_1(x) = (a_1+\epsilon_1)x,$

$K_1(x) = -\epsilon_1 x - (1-a_1)\frac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) - (1-a_1)\int_\eta^t \int_0^s h(\tau)f(\tau, x(\tau).x'(\tau))d\tau ds,$

$L_1(x) = P_1(x) + K_1(x).$

Our first observation is

Lemma 2.1. *Let $x(t)$ be a fixed point of the operator L_1. Then $x(t)$ is a solution to the problem* (1.1).

Proof. Since $x(t)$ is a fixed point of the operator L_1 then

$$\begin{aligned} x(t) &= L_1(x) = P_1(x) + K_1(x) \\ &= (a_1+\epsilon_1)x(t) - \epsilon_1 x(t) - (1-a_1)\tfrac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) \\ &-(1-a_1)\textstyle\int_\eta^t \int_0^s h(\tau)f(\tau,x(\tau),x'(\tau))d\tau \\ &= a_1 x(t) - (1-a_1)\tfrac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) \\ &-(1-a_1)\textstyle\int_\eta^t \int_0^s h(\tau)f(\tau,x(\tau),x'(\tau))d\tau, \end{aligned}$$

from here

$$(1-a_1)x(t) = -(1-a_1)\frac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) - (1-a_1)\int_\eta^t \int_0^s h(\tau)f(\tau,x(\tau),x'(\tau))d\tau$$

Ref.

[1] Zhao, J., F. Geng, J. Zhao, W. Ge. Positive solutions for a new kind Sturm - Liouville like foour point boundary value problem, Applied Mathematics and Computations, 2010, pp.811-819.

and

$$x(t) = -\frac{x'(1)}{\alpha_2} + \alpha_1(t-\eta)x(\xi) - \int_\eta^t \int_0^s h(\tau)f(\tau, x(\tau), x'(\tau))d\tau \tag{2.2}$$

Now we differentiate the last equality with respect t and we obtain

$$x'(t) = \alpha_1 x(\xi) - \int_0^t h(\tau)f(\tau, x(\tau), x'(\tau))d\tau, \tag{2.3}$$

again we differentiate the last equality with respect t and we have

$$x''(t) = -h(t)f(t, x(t), x'(t)).$$

We put $t = 0$ in (2.3) and we obtain

$$x'(0) = \alpha_1 x(\xi),$$

after we put $t = \eta$ in (2.2) we get

$$x(\eta) = -\frac{x'(1)}{\alpha_2},$$

therefore $x(t)$ satisfies the problem (1.1).

The above Lemma motivate us to search fixed points of the operator L_1. For this purpose we will use the following fixed point theorem.

Theorem 2.2. *(see [2], Corrolary 2.4, pp. 3231) Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*
(i) *S is continuous, $S(X)$ resides in a compact subset of Y;*
(ii) *$T : X \longrightarrow Y$ is expansive and onto.*
Then there exists a point $x^\star \in X$ with $Sx^\star + Tx^\star = x^\star$.

Here we will use the following definition for expansive operator.

Definition. (see [2], pp. 3230) Let (X, d) be a metric space and M be a subset of X. The mapping $T : M \longrightarrow X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y) \quad \forall x, y \in M.$$

Lemma 2.3. *The operator $P_1 : N_1 \longrightarrow N_1^*$ is an expansive operator and onto.*

Proof. Let $x(t) \in N_1$. Then $x(t) \in \mathcal{C}^1([0, 1])$, $|x(t)| \leq D_1$, $|x'(t)| \leq D_1$, from here $P_1(x) \in \mathcal{C}^1([0, 1])$ and $|P_1(x)| \leq (a_1 + \epsilon_1)D_1$, $\left|\frac{d}{dt}P_1(x)\right| \leq (a_1 + \epsilon_1)D_1$, i.e. $P_1(x) \in N_1^*$ and $P_1 : N_1 \longrightarrow N_1^*$.

Let $x, y \in N_1$. Then

$$||P_1(x) - P_1(y)|| = (a_1 + \epsilon_1)||x - y||,$$

consequently $P_1 : N_1 \longrightarrow N_1^*$ is an expansive operator with a constant $h = a_1 + \epsilon_1 > 1$.

Let now $y \in N_1^*$, $y \neq 0$. Then $x = \frac{y}{a_1+\epsilon_1} \in N_1$ and $P_1(x) = y$, then $P_1 : N_1 \longrightarrow N_1^*$ is onto.

Lemma 2.4. *The operator $K_1 : N_1 \longrightarrow N_1$ is a continuous operator.*

Proof. Let $x(t) \in N_1$. Then $K_1(x) \in \mathcal{C}^1([0, 1])$ and

$$|K_1(x)| \leq \epsilon_1|x| + (1-a_1)\frac{|x'(1)|}{|\alpha_2|} + (1-a_1)|\alpha_1||x(\xi)| + (1-a_1)\int_\eta^t \int_0^s |h(\tau)||f(\tau, x(\tau), x'(\tau)|d\tau$$

Ref.

[2] Xiang, T., Rong Yuan. A class of expansive - type Krasnosel'skii fixed point theorems. Nonlinear Analysis, 71(2009), 3229-3239.

$$\leq \epsilon_1 D_1 + (1-a_1)\frac{D_1}{|\alpha_2|} + (1-a_1)|\alpha_1|D_1 + (1-a_1)M_1 \leq D_1,$$

in the last inequality we use the first inequality of (2.1), also

$$\left|\frac{d}{dt}K_1(x)\right| \leq \epsilon_1|x'(t)| + (1-a_1)|\alpha_1||x(\xi)| + (1-a_1)\int_0^t |h(\tau)||f(\tau, x(\tau), x'(\tau)|d\tau$$

$$\leq \epsilon_1 D_1 + (1-a_1)|\alpha_1|D_1 + (1-a_1)M_1 \leq D_1,$$

in the last inequality we use the second inequality of (2.1). Therefore

$$K_1 : N_1 \longrightarrow N_1.$$

Since h and f are continuous functions from $x_n \longrightarrow_{n\longrightarrow\infty} x$, $x_n, x \in N_1$, in the sense of the topology of the set N_1 we have $K_1(x_n) \longrightarrow_{n\longrightarrow\infty} K_1(x)$ in the sense of the topology of the set N_1, in other words the operator $K_1 : N_1 \longrightarrow N_1$ is a continuous operator.

From Lemma 2.1, Theorem 2.2, Lemma 2.3 and Lemma 2.4 follows that the operator L_1 has a fixed point $x^1 \in N_1^*$ which is a solution to the problem (1.1). From (2.3), since f and h are continuous functions, follows that $x^1(t) \in \mathcal{C}^2([0,1])$.

2) If we suppose that the bounded solution $x^1(t) \equiv 0$. Then, from (2.2), we have

$$\int_\eta^t \int_0^s h(\tau)f(\tau, x(\tau), x^{1'}(\tau))d\tau ds = 0 \quad \forall t \in [0,1],$$

which is a contradiction.

3) If we suppose that the bounded solution $x(t)$ coincides with a constant, then from the equation of the problem (1.1) we conclude that

$$h(t)f(t, x^1(t), x^{1'}(t)) = 0 \quad \forall t \in [0,1],$$

which is a contradiction.

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[2] Xiang, T., Rong Yuan. A class of expansive - type Krasnosel'skii fixed point theorems. Nonlinear Analysis, 71(2009), 3229-3239.

Notes